TOPICS IN STATISTICAL PHYSICS AND PROBABILITY THEORY
HOMEWORK SHEET 1

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To hand in by April 25 to the instructor in class.

(i) Denote the entropy function $H : [0, 1] \to [0, \infty)$ by

$$H(x) := -x \log x - (1 - x) \log(1 - x), \tag{1}$$

where the logarithms are in base e. Prove that for any integers $n \geqslant k \geqslant 0$,

$$\frac{1}{n+1} e^{nH(\frac{k}{n})} \leqslant \binom{n}{k} \leqslant e^{nH(\frac{k}{n})}.$$

Hint: Instead of resorting to Stirling's approximation, a neat proof is obtained by considering the binomial distribution $\text{Bin}\left(n, \frac{k}{n}\right)$.

(ii) (Curie-Weiss model) Let $\beta \geqslant 0$ and $h \in \mathbb{R}$. Recall the *limiting rate function* for the magnetization density in the Curie-Weiss model, the function $\varphi_{\beta,h} : [-1, 1] \to \mathbb{R}$ defined by

$$\varphi_{\beta,h}(m) := \frac{1}{2}\beta m^2 + hm + H\left(\frac{1+m}{2}\right),$$

where H is given in (1).

(a) Prove that $\varphi_{\beta,h}$ attains its global maximum at a *unique* point $m^* \in [-1, 1]$ in the case that $\beta \leqslant 1$ or $h \neq 0$. In addition, show that $m^* = 0$ when $\beta \leqslant 1$ and $h = 0$.

(b) Prove that $\varphi_{\beta,h}$ attains its global maximum at exactly two points $\pm m^*$ with $m^* \in (0, 1]$ when $\beta > 1$ and $h = 0$. In addition, show that

$$\lim_{\beta \downarrow 1} \frac{m^*}{\sqrt{3(\beta - 1)}} = 1$$

Remark: The exponent $\frac{1}{2}$ of $\beta - 1$ is called a critical exponent as it measures how the magnetization density behaves in the vicinity of the critical point.

(iii) (One-dimensional Ising model). Let $n \geqslant 2$ and $f : \{1, 2, \ldots, n\} \to \{-1, 1\}$ be a random function sampled according to the one-dimensional Ising model at inverse temperature $\beta > 0$ and magnetic field $h \in \mathbb{R}$. That is,

$$\mathbb{P}(f) = \frac{1}{Z_{\beta,h,n}} \exp\left(\beta \sum_{i=1}^{n-1} f(i)f(i+1) + h \sum_{i=1}^{n} f(i)\right),$$

where $Z_{\beta,h,n}$ is the partition function (which normalizes the above expression to be a probability measure).

(a) Prove that there exist $c_1(\beta, h), c_2(\beta, h)$, analytic functions on $\beta > 0, h \in \mathbb{R}$, so that

$$Z_{\beta,h,n} = c_1(\beta, h)\lambda_+^n + c_2(\beta, h)\lambda_-^n$$

with

$$\lambda_\pm = e^\beta \cosh(h) \pm \sqrt{e^{2\beta} \cosh^2(h) - 2\sinh(2\beta)}.$$

Conclude that

$$\lim_{n\to\infty} \frac{1}{n} \log(Z_{\beta,h,n}) = \log(\lambda_+). \tag{2}$$

Remark: The limit on the left-hand side of (2) is called the *pressure* of the model.
Hint: One can use a *transfer matrix approach* (an approach related to linear recursion relations or Markov chain theory): relate Z to the n'th power of certain 2×2 matrix.

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(b) Observe that the magnetization density satisfies

$$\mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n} f(i)\right) = \frac{1}{n}\cdot\frac{d}{dh}\log(Z_{b,h,n})$$

and deduce that the limiting magnetization density,

$$\lim_{n\to\infty}\mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n} f(i)\right)$$

exists and is an analytic function of β, h in the entire regime $\beta > 0$, $h \in \mathbb{R}$. In other words, there is no spontaneous magnetization in the one-dimensional Ising model.

(c) Another manifestation of the lack of spontaneous magnetization is the fact that

$$\lim_{n\to\infty}\mathbb{E}\left(\frac{1}{n^2}\left(\sum_{i=1}^{n} f(i)\right)^2\right) = 0 \quad \text{when } h = 0, \text{ for all } \beta \geqslant 0.$$

Deduce this from part (a) by first showing that

$$\mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^{n} f(i)\right) = \frac{1}{n^2}\cdot\frac{d^2}{dh^2}\log(Z_{b,h,n}) \quad \text{for all } \beta > 0,\ h \in \mathbb{R}.$$

(iv) (Star-triangle (Yang-Baxter) transformation). Consider a ferromagnetic Ising model on a general finite graph $G = (V(G), E(G))$ at inverse temperature $\beta \geqslant 0$ and zero magnetic field. Precisely, the probability of each configuration $f : V(G) \to \{-1, 1\}$ is given by

$$\mathbb{P}(f) = \frac{1}{Z_{\beta,G}}\exp\left(\beta\sum_{\{u,v\}\in E(G)} f(u)f(v)\right).$$

Suppose that $v_0 \in V(G)$ has degree 3 and denote its neighbors by $u_1, u_2, u_3 \in V(G)$. Denote by g the restriction of the function f to the vertex set $V(G)\setminus\{v_0\}$. Prove that the (marginal) distribution of g is given by

$$\mathbb{P}(g) = \frac{1}{Z'_{\beta,G}}\exp\left(\beta\sum_{\{u,v\}\in E'(G)} g(u)g(v) + \gamma\left(g(u_2)g(u_3) + g(u_1)g(u_3) + g(u_1)g(u_2)\right)\right)$$

for some $Z'_{\beta,G}$, where $E'(G) = E(G)\setminus\{\{u_1, v_0\}, \{u_2, v_0\}, \{u_3, v_0\}\}$ and

$$\gamma := \frac{1}{4}\log\left(e^{2\beta} + e^{-2\beta} - 1\right). \tag{3}$$

In other words, the restriction of f to $V(G)\setminus\{v_0\}$ is still an Ising model, on the graph G with vertex v_0 and its three adjoining edges (forming a 'star') removed and with a 'triangle' of edges added on the neighbors of v_0, on which the coupling constant is changed from β to γ.

Remark: A similar procedure applies when each edge e is given its own coupling constant $\beta_e \geqslant 0$. In particular, suppose we start with an Ising model at inverse temperature $\beta \geqslant 0$ on a piece of the hexagonal lattice. By restricting the model to one bipartition class of the lattice we may obtain an Ising model at inverse temperature γ given by (3) on a piece of the triangular lattice.

(v) (Connectivity of boundaries following Timár 2013. This is an **optional exercise**).

Definitions: A graph is *locally finite* if all degrees are finite. A graph is *even* if the degrees of all its vertices are even. The cycle space of a graph $G = (V, E)$ is the vector space over $\mathbb{F}_2$ of all spanning even subgraphs of G (regarded as vectors in $\{0, 1\}^E$). A *separating set* is a set of edges $\Pi \subset E$ for which there exist two vertices $x, y \in V$ such that every path between x and y intersects Π. A separating set is said to be *minimal* if it is minimal with respect to inclusion.

Let $G = (V, E)$ be a locally finite connected graph, let Π be a minimal separating set in G and let $\mathcal{C}$ be a set of cycles in G which generate the cycle space of G (every cycle can be written as a linear combination over $\mathbb{F}_2$ of the cycles in $\mathcal{C}$).

(a) Show that Π splits G into two components, i.e., that the graph $(V, E \setminus \Pi)$ has exactly two connected components.
(b) Let $\{\Pi_1, \Pi_2\}$ be a non-trivial partition of Π. Show that there exists a cycle $c \in \mathcal{C}$ which intersects both Π_1 and Π_2. (Hint: find two paths P_1 and P_2 between some x and y, such that P_i does not intersect Π_i, decompose their sum in the cycle space, and use parity considerations).
(c) Let $A \subset V$ be such that both A and $V \setminus A$ are non-empty and connected. Show that the edge boundary $\partial A := \{\{u, v\} \in E : u \in A, v \notin A\}$ of A is a minimal separating set.
(d) Let $G^* = (V, E^*)$ be a locally finite graph on the same vertex set as G and assume that every element in $\mathcal{C}$ is a clique in G^*. Denote the internal vertex boundary of a set $A \subset V$ (in the graph G) by

$$\partial_{\text{in}} A := \big\{u \in A : \{u, v\} \in E \text{ for some } v \in V \setminus A\big\}.$$

Show that if both A and $V \setminus A$ are connected in G, then $\partial_{\text{in}} A$ is connected in G^*. (Hint: assume that the vertex boundary is not connected and construct from it a non-trivial partition of the edge boundary).
(e) Deduce the claim stated in class for $\mathbb{Z}^d$. Namely, if $A \subset \mathbb{Z}^d$ is a finite connected set such that A^c is connected, then $\partial_{\text{in}} A$ is connected in the graph $(\mathbb{Z}^d)^{\boxtimes}$ obtained from $\mathbb{Z}^d$ by adding edges of the form $\{x, x \pm e_i \pm e_j\}$, where $x \in \mathbb{Z}^d$ and $1 \leqslant i < j \leqslant d$. (The main issue here is proving that the set of basic 4-cycles generates the cycle space of $\mathbb{Z}^d$).